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HIDDEN SYMMETRIES AND DECAY FOR THE VLASOV EQUATION ON THE KERR SPACETIME

L. ANDERSSON, P. BLUE, AND J. JOUDIOUX

ABSTRACT. This paper proves the existence of a bounded energy and integrated energy decay for solutions of the massless Vlasov equation in the exterior of a very slowly rotating Kerr spacetime. This combines methods previously developed to prove similar results for the wave equation on the exterior of a very slowly rotating Kerr spacetime with recent work applying the vector-field method to the relativistic Vlasov equation.

1. INTRODUCTION

In this paper we prove the existence of a bounded energy, and an integrated energy decay estimate for solutions of the massless Vlasov equation in the exterior of a very slowly rotating Kerr spacetime.

For parameters a, M , with $|a| \leq M$, the exterior region of the Kerr spacetime is represented in Boyer-Lindquist coordinates (t, r, θ, ϕ) by $\mathbb{R} \times (r_+, \infty) \times S^2$ with the Lorentzian metric

$$(1.1) \quad g = - \left(1 - \frac{2Mr}{\Sigma} \right) dt^2 - \frac{4Mar \sin^2 \theta}{\Sigma} dt d\phi + \frac{\Pi \sin^2 \theta}{\Sigma} d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2,$$

where $r_+ = M + \sqrt{M^2 - a^2}$, and

$$\Delta = r^2 - 2Mr + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad \Pi = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta.$$

For $|a| \leq M$, the Kerr spacetimes contain a black hole and are stationary and axisymmetric, that is to say ∂_t and ∂_ϕ are Killing vector fields. Although the exterior is extendible as an analytic manifold, it is globally hyperbolic and foliated by surfaces of constant t , Σ_t , which are Cauchy surfaces.

The Vlasov equation governs the evolution of massive or massless particles which do not self-interact [14]. The particles are represented by a distribution function on phase space, which evolves under the geodesic flow, so it is constant along geodesics. In the context of kinetic theory, the equation is known as the collisionless Boltzmann equation.

Let (\mathcal{M}, g) be a time oriented Lorentzian manifold of dimension $1 + 3$, with timelike vector field T_+ . For the case of massless Vlasov, the distribution function is a non-negative function defined on the bundle of future light cones \mathcal{C}^+ ,

$$\begin{aligned} \mathcal{C}^+ &= \bigcup_{x \in \mathcal{M}} \mathcal{C}_x^+, \\ \mathcal{C}_x^+ &= \{(x, v) : v \in T_x \mathcal{M}, g(v, v) = 0, g(v, T_+) < 0\}. \end{aligned}$$

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For $m > 0$, the set $g(v, v) = -m^2$, $g(v, T_+) < 0$ is sometimes called the mass shell, and \mathcal{C}^+ is its analogue for the massless case considered here. The Vlasov equation is

$$(1.2) \quad \mathcal{X}f = 0,$$

where \mathcal{X} is the geodesic spray, the vector field on $T\mathcal{M}$ which generates the geodesic flow. The geodesic spray is the Lagrangian vector field of $L = \frac{1}{2}g(v, v)$ [1, §3.7]. We have that $\mathcal{X}L = 0$, in particular \mathcal{X} is tangent to \mathcal{C}^+ . In case the distribution function f is a function $f : \mathcal{C}^+ \rightarrow \mathbb{R}$, we shall refer to equation (1.2) as the massless Vlasov equation.

A local coordinate system (x^a) on \mathcal{M} induces natural coordinates (x^a, v^a) on $T\mathcal{M}$, where $v^a = dx^a(v)$. The coordinate form of \mathcal{X} is

$$(1.3) \quad \mathcal{X} = v^a \left(\frac{\partial}{\partial x^a} - v^b \Gamma_{ab}^c \frac{\partial}{\partial v^c} \right),$$

where Γ_{ab}^c is the Christoffel symbol of the metric g_{ab} .

In the Kerr exterior, it is convenient to use the Boyer-Lindquist coordinates $(x^a) = (t, r, \theta, \phi)$ and the corresponding natural coordinates (x^a, v^a) . On \mathcal{C}^+ , we locally use coordinates $(t, r, \theta, \phi, v^r, v^\theta, v^\phi)$, and treat the quantities $v^t, v_t, v_r, v_\theta, v_\phi$ as functions of these. To facilitate the presentation of our main result, we introduce

$$E_{\text{model},3}[f](t) = \int_{\Sigma_t} \int_{\mathcal{C}_x^+} \left(\frac{(r^2 + a^2)^2}{\Delta} v_t^2 + \Delta v_r^2 + v_\theta^2 + \frac{1}{\sin^2 \theta} v_\phi^2 \right) |f|_2 d^3v d^3x,$$

where

$$|f|_2 = \left| M^2 v_t^2 + v_\theta^2 + \frac{1}{\sin^2 \theta} v_\phi^2 \right|^2 f, \quad d^3x = \sin \theta dr d\theta d\phi, \quad d^3v = \frac{1}{|v_t|} r^2 \sin \theta dv^r dv^\theta dv^\phi.$$

The term $|f|_2$ should be understood as a strengthening of the f by two factors of $M^2 v_t^2 + v_\theta^2 + \sin^{-2} \theta v_\phi^2$. As explained in Section 3.3, these two factors arise from strengthening the energy by two second-order multiplication symmetries of the Vlasov equation. The volume forms d^3x and d^3v are given here because they have simple coordinate expressions, although they are not the naturally induced volume forms defined on Σ_t and \mathcal{C}_x^+ , which are used in the rest of this paper and introduced in Sections 2.1 and 2.2.

Our main results are

Theorem 1 (Uniformly bounded energy). *There are positive constants C and $\bar{\epsilon}$ such that if $M > 0$, $|a| \leq \bar{\epsilon}M$, and $f : \mathcal{C}^+ \rightarrow [0, \infty)$ is a smooth solution of the Vlasov equation (1.2) in the exterior of the Kerr spacetime with parameters (M, a) , then, for all t in \mathbb{R} ,*

$$(1.4) \quad E_{\text{model},3}[f](t) \leq C E_{\text{model},3}[f](0).$$

Theorem 2 (Morawetz estimate). *There are positive constants C , $\bar{\epsilon}$, and \bar{r} and a function $\mathbf{1}_{r \neq 3M}$ which is identically 1 for $|r - 3M| \geq \bar{r}$ and zero otherwise such that if $M > 0$, $|a| \leq \bar{\epsilon}M$, and $f : \mathcal{C}^+ \rightarrow [0, \infty)$ is a smooth solution of the Vlasov equation (1.2) in the exterior of the Kerr spacetime with parameters (M, a) , then,*

$$(1.5) \quad \int_{-\infty}^{\infty} \int_{\Sigma_t} \int_{\mathcal{C}_x^+} \left(\left(M \frac{\Delta^2}{(r^2 + a^2)^2} \right) v_r^2 + \mathbf{1}_{r \neq 3M} \frac{1}{r} \left(M^2 v_t^2 + v_\theta^2 + \frac{1}{\sin^2 \theta} v_\phi^2 \right) \right) |f|_2 d^3v d^4x, \\ \leq C E_{\text{model},3}[f](0),$$

where $d^4x = d\bar{t}d^3x$.

More precisely,

$$(1.6) \quad \int_{-\infty}^{\infty} \int_{\Sigma_t} \int_{\mathcal{C}_x^+} M \frac{\Delta^2}{(r^2 + a^2)^2} v_r^2 |f|_2 + r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L} f d\mu_{\mathcal{C}_x^+} d\mu_g \leq C E_{\text{model},3}[f](0),$$

where $\tilde{\mathcal{R}}'$ is given in equation (3.9) and where $d\mu_{\mathcal{C}_x^+}$ and $d\mu_g$ are the natural volume forms on \mathcal{C}_x^+ and \mathcal{M} by the metric g .

The main innovation in this paper is to combine the vector field technique introduced in [13] for proving dispersive estimate for the relativistic Vlasov equation with earlier work on dispersion of fields outside a Kerr black hole, in particular the method of [3], see also [2].

The method used in [3] is a generalization of the vector-field method, which relies on the stress-energy tensor and spacetime symmetries to construct momenta appropriate for the proof of energy estimates and integrated energy estimates. The proof of the non-linear stability of Minkowski space [9] is an important application of this method. The vector-field method was recently applied to prove dispersive estimates for the relativistic Vlasov equation as part of a proof of non-linear stability for the massless and massive Vlasov-Nordström systems on Minkowski space [13] (see also [17] for the nonrelativistic Vlasov equation). Previous stability results for Minkowski space as a solution of the Einstein-Vlasov system include the massive [15] and massless Vlasov cases [10] in spherical symmetry and, recently, the massless case without symmetry [19].

Energy bounds and Morawetz estimates, analogous to Theorems 1 and 2 respectively, have already been proved for the wave equation outside a very slowly rotating Kerr black hole [11, 18, 3]. Strictly speaking, an energy bound should be an integral over spacelike hypersurfaces of an integrand that is quadratic in v , rather than of order 6, as appears in $E_{\text{model},3}[f]$, but we will consistently ignore this distinction. Away from (an open set about) $r = 3M$, the horizon at $r = r_+$, and null infinity at $r \rightarrow \infty$, the integrand in the Morawetz estimate is a bounded multiple of the integrand appearing in the energy; however, the integral is over all of space-time, instead of a single spacelike hypersurface. Thus, the Morawetz estimate implies that the local energy in a fixed r region (away from $r = r_+$ and $r \rightarrow \infty$ and sufficiently far from $r = 3M$) is integrable in time. Hence, on average, it must decay in time. Thus, Morawetz estimates are also called integrated local energy decay estimates. Energy bounds and Morawetz estimates are a useful tool in proving pointwise estimates, for instance of the form $\sup_{r \in (r_+, R], (\theta, \phi) \in S^2} |\psi(t, r, \theta, \phi)| \lesssim t^{-p}$ for some p . For the wave equation in the subextremal range $|a| < M$, the entire argument from energy estimates and Morawetz bounds to pointwise bounds can be found in [12].

In the Kerr spacetime, there are null geodesics that can orbit at fixed r , and these are the primary obstacle in proving Morawetz estimates. Furthermore, for $|a| > 0$, the vector field ∂_t ceases to be timelike near $r = r_+$, which prevents the existence of a conserved, positive energy. Since both the wave equation and massless Vlasov equation admit solutions that approximate null geodesics for arbitrary lengths of time, any Morawetz estimate must degenerate on such solutions. On the orbiting null geodesics, the factor $\tilde{\mathcal{R}}'$ vanishes, providing sufficient degeneracy; for $|a| \ll M$, the roots of $\tilde{\mathcal{R}}'$ are all near $r = 3M$, which is why $\tilde{\mathcal{R}}'^2$ in equation (1.6) can be replaced by $\mathbf{1}_{r \neq 3M} (M^2 v_t^2 v_\theta^2 + \sin^{-2} \theta v_\phi^2)^2$.

in equation (1.5). Our analysis is dependent on the fact that $\tilde{\mathcal{R}}'$ can be expanded purely in terms of r dependent factors and constants of motion along the null geodesics. This is a consequence of the remarkable observation of Carter that, in addition to the geodesic constants of motion arising from the metric and the two Killing vectors, there is a fourth constant of motion, called a hidden symmetry [8].

Steady states for the massive Vlasov equation in the exterior of a fixed Schwarzschild space-time (which has $a = 0$ and represents the exterior of a star or black hole) have been constructed and used to study accretion disks [16]. The existence of these steady states implies that no Morawetz estimate, analogous to Theorem 2, can hold for the massive Vlasov equation outside a Schwarzschild black hole.

The formation of black holes for the massive Einstein-Vlasov system has been studied in [4, 6]. For the coupled Einstein-massless Vlasov system, there are spherically symmetric steady states [5]. The existence of such solutions suggests, in contrast to the results in this paper, that there are spherically symmetric solutions of the Einstein-massless Vlasov system which have a nonzero, static configuration of massless Vlasov matter outside a Schwarzschild-like black hole. However such solutions seem to require a large Vlasov field and cannot be small perturbations of the Schwarzschild solution.

The paper is organized as follows. Section 2 contains an introduction to the geometry of the Kerr spacetime, and in particular a discussion on the multiplication symmetries of the field in Section 2.1, and a presentation of the properties of the stress-energy tensor of the Vlasov equation in Section 2.2. Section 3 contains the proof of the Morawetz estimates. The relevant energies are defined in Section 3.2; Section 3.3 introduces the vector field used to perform the estimates; relevant bulk terms are estimated in Section 3.4; and the proof is concluded in Section 3.5.

2. PRELIMINARIES

Throughout, the indices $\alpha, \beta, \gamma, \dots$ denote integers in $\{0, 1, 2, 3\}$. Underlined indices $\underline{a}, \underline{b}, \underline{c}, \dots$ are used exclusively to parametrize the set of symmetries used for the calculation, as explained in Section 2.1. The Einstein summation convention is used throughout the paper.

2.1. The Kerr geometry. For $M > 0$ and $|a| \leq M$, the exterior region of the Kerr space-time is $(t, r, \omega) \in \mathbb{R} \times (r_+, \infty) \times S^2$, where $r_+ = M + \sqrt{M^2 - a^2}$ is the larger of the two roots of $\Delta = 0$, together with the metric given in equation (1.1). Typically, we will parametrise S^2 by spherical coordinates θ, ϕ . Although this exterior can be extended as a smooth Lorentzian manifold, it is globally hyperbolic, with the surfaces of constant t providing a foliation by Cauchy hypersurfaces. Thus, there is a well-posed initial-value-problem for many PDEs, including the Vlasov equation, with initial data posed, for example, on the hypersurface $t = 0$. For $M > 0$ and $|a| \leq M$, the exterior region of the Kerr space-time describes the exterior region of a rotating black hole.

The vector field $T_\perp = \partial_t + \omega_\perp \partial_\phi$ with $\omega_\perp = 2aMr/\Pi$ is orthogonal to the surfaces of constant t , and hence to $\partial_r, \partial_\theta, \partial_\phi$. This vector field is not normalised, and, instead, $g(T_\perp, T_\perp) = -\Delta\Sigma/\Pi$. The rotation speed of the black hole is $\omega_{\mathcal{H}} = a/(r_+^2 + a^2)$. Independently of θ , one has $\omega_{\mathcal{H}} = \lim_{r \rightarrow r_+} \omega_\perp$.

Many calculations are simplified by working only with the following form of the inverse Kerr metric:

$$(2.1) \quad \Sigma g^{\alpha\beta} = \Delta \partial_r^\alpha \partial_r^\beta + \frac{1}{\Delta} \mathcal{R}^{\alpha\beta},$$

where

$$\begin{aligned} \Delta &= r^2 - 2Mr + a^2, \\ \Sigma &= \Omega^{-2} = r^2 + a^2 \cos^2 \theta, \\ \mathcal{R}^{\alpha\beta} &= -(r^2 + a^2)^2 \partial_t^\alpha \partial_t^\beta - 4aMr \partial_t^{(\alpha} \partial_\phi^{\beta)} + (\Delta - a^2) \partial_\phi^\alpha \partial_\phi^\beta + \Delta Q^{\alpha\beta}, \\ Q^{\alpha\beta} &= \partial_\theta^\alpha \partial_\theta^\beta + \cot^2 \theta \partial_\phi^\alpha \partial_\phi^\beta + a^2 \sin^2 \theta \partial_t^\alpha \partial_t^\beta. \end{aligned}$$

This form of the expression allows us to avoid having to work with $\Pi = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta$, except when working with $T_\perp = \partial_t + (2aMr/\Pi) \partial_\phi$. In fact, except in the volume form and inside $Q^{\alpha\beta}$, this notation typically allows us to avoid all θ dependent factors. It will later be useful to use a conformal factor Ω defined by

$$\Omega^{-2} = \Sigma.$$

The volume form on the Kerr exterior in Boyer-Lindquist coordinates is

$$\sqrt{|g|} dt dr d\theta d\phi = \Sigma \sin \theta dt dr d\theta d\phi.$$

Since Σ is uniformly equivalent to r^2 , integrals with respect to this volume form are equivalent to those with respect to $d^3x dt$.

In the Kerr spacetime, ∂_t and ∂_ϕ are Killing vectors and $Q^{\alpha\beta}$ is a conformal Killing tensor. We use the following notation

$$e = v_\alpha \partial_t^\alpha, \quad l_z = v_\alpha \partial_\phi^\alpha, \quad q = v_\alpha v_\beta Q^{\alpha\beta}.$$

The quantities e, l_z, q are multiplicative symmetries of the null Vlasov equation (see Definition 4 for the precise definition). Further symmetries can be constructed as polynomials of e, l_z, q . Of particular importance in our analysis is $\mathbb{S}_2 = \{e^2, e l_z, l_z^2, q\}$, which is a set of four symmetries that are quadratic in v . We index this four-element set by underlined, lower-case Roman indices, e.g. \underline{a} , which we call symmetry indices, and denote the corresponding element of \mathbb{S}_2 by $S_{\underline{a}}$. This allows us to write

$$\mathbb{S}_2 = \{e^2, e l_z, l_z^2, q\} = \{S_{\underline{a}}\}.$$

The symmetry index should not be confused with (abstract or concrete) space-time indices given by lower-case Greek indices, e.g. α . Since each element of \mathbb{S}_2 is the contraction of a 2-tensor with $v_\alpha v_\beta$, we can introduce a four-element set of tensors $\{S_{\underline{a}}^{\alpha\beta}\}$ such that, for each \underline{a} ,

$$S_{\underline{a}} = S_{\underline{a}}^{\alpha\beta} v_\alpha v_\beta.$$

For specificity, one can take the symmetry indices to range over the set \mathbb{S}_2 itself and to take $S_{e^2} = e^2$, $S_{e^2}^{\alpha\beta} = \partial_t^\alpha \partial_t^\beta$, and so on.

The quantity $\mathcal{R}^{\alpha\beta} v_\alpha v_\beta$ plays a crucial role in our analysis. It can be written as a linear combination of the $S_{\underline{a}}$, with coefficients that are polynomial in r, M, a . To simplify a lot of the calculations, we use the notation $\mathcal{R}^{\underline{a}}$ to denote these coefficients and use the

Einstein summation convention in the \underline{a} indices. We also use the notation \mathcal{R} to denote $\mathcal{R}^{\alpha\beta}v_\alpha v_\beta$. Thus, we have four expressions for the following quantity

$$(2.2) \quad \mathcal{R} = \mathcal{R}^{\alpha\beta}v_\alpha v_\beta = \mathcal{R}^{\underline{a}}S_{\underline{a}} = \mathcal{R}^{\underline{a}}S_{\underline{a}}^{\alpha\beta}v_\alpha v_\beta.$$

For other quantities that are linear combinations of the $S_{\underline{a}}$ (possibly with coefficients that are polynomial or rational functions of r, M, a), we also use the Einstein summation convention in the \underline{a} variables to expand the quantity. In addition to \mathcal{R} and derivatives of rescalings of it, we also use

$$\mathcal{L} = M^2 e^2 + l_z^2 + q,$$

which can be expanded as

$$(2.3) \quad \mathcal{L}^{\underline{a}}S_{\underline{a}} = \mathcal{L}^{\underline{a}}S_{\underline{a}}^{\alpha\beta}v_\alpha v_\beta.$$

One crucial way in which \mathcal{R} appears is the analysis of whether null geodesics fall into the black hole, asymptote to an orbiting null geodesic, or escape to infinity. Equation (2.1) can be contracted with $v_\alpha v_\beta$ to derive an equation for $dr/d\lambda$. One consequence of this is that a null geodesic has a turning point, where $dr/d\lambda$ vanishes, only when $\mathcal{R} = 0$. Furthermore, by a standard dynamical systems analysis of one-dimensional systems, there can only be a trajectory remaining at fixed r when $\mathcal{R} = 0 = \partial_r \mathcal{R}$. Null geodesics remaining at fixed r are of great importance in the study of fields outside black holes, and, for convenience, we call them orbits. Outside a nonrotating Schwarzschild black hole, the orbits are only at $r = 3M$, but when $a \neq 0$, the orbits bifurcate to a range of r values.

Following [3], in the remainder of the paper, it is useful to consider collections of tensor fields that are indexed by a pair of symmetry indices, e.g. $\{X^{\alpha\beta}\partial_\alpha\}$. A collection of vector fields and a collection of symmetric 2-tensors can be contracted against each other, in the same way as a vector field and a stress-energy tensor can be contracted. This gives an energy, and the change in this energy can be estimated using methods very similar to those for energies arising from vector fields and stress-energy tensors, see 3.1. For this reason, we use the terminology “symmetry-strengthened vector field” to more concisely refer to collections of vector fields that are indexed by sets of symmetry indices. We wish to point out that a symmetry-strengthened vector field is a collection of vector fields and not a vector field. Similarly, we use “symmetry-strengthened stress-energy tensor” for a collection of symmetric 2-tensors indexed by symmetry indices.

2.2. The stress-energy tensor and symmetries of the Vlasov equation. Throughout this subsection, let (\mathcal{M}, g) be a globally hyperbolic, Lorentzian manifold of dimension $3 + 1$. Consider the vector bundle $\mathcal{V} = T\mathcal{M}$, and consider \mathcal{C}^+ .

The volume element on \mathcal{C}_x^+ induced from the volume element $d\mu_{T_x\mathcal{M}} = (-\det g)^{1/2}dv^0 \wedge \dots \wedge dv^3$ is given by the Gelfand-Leray form [7, Chapter 7] of $d\mu_{T_x\mathcal{M}}$ with respect to $-L = -\frac{1}{2}g(v, v)$, restricted to \mathcal{C}_x^+ . That is $d\mu_{T_x\mathcal{M}} = dL \wedge d\mu_{\mathcal{C}_x^+}$, where dL is the exterior derivative of L calculated on $T_x\mathcal{M}$. In local coordinates (x^a) with a taking values $0, \dots, 3$, this takes the form

$$d\mu_{\mathcal{C}_x^+} = \sqrt{|g|} \frac{dv^1 \wedge dv^2 \wedge dv^3}{(-v_0)}.$$

This can also be found as an appropriate limit of $i_X(\mathrm{d}\mu_{T_x\mathcal{M}})$ with $X^\alpha = (-g(v, v))^{-1}v^\alpha$ on the hypersurface $\{v : g(v, v) = -m^2, \text{ and } v \text{ future directed}\}$ as $m \rightarrow 0^+$.

In the particular case of the Kerr spacetime, recall $\sqrt{|g|} = \Sigma \sin \theta$ is uniformly equivalent to $r^2 \sin \theta$. The volume form fixing the overall orientation of the tangent space to the Kerr exterior is

$$\mathrm{d}\mu_{T_x\mathcal{M}} = \sqrt{|g|} \mathrm{d}v^t \wedge \mathrm{d}v^r \wedge \mathrm{d}v^\theta \wedge \mathrm{d}v^\phi.$$

The differential of the function L is given by

$$\mathrm{d}L = -v^\alpha g_{\alpha t} \mathrm{d}v^t - v^\alpha g_{\alpha r} \mathrm{d}v^r - v^\alpha g_{\alpha \theta} \mathrm{d}v^\theta - v^\alpha g_{\alpha \phi} \mathrm{d}v^\phi.$$

Hence, the factorization of the volume on $T_x\mathcal{M}$ is then given by

$$\mathrm{d}\mu_{T_x\mathcal{M}} = -v_0 \mathrm{d}v^0 \wedge \underbrace{\left(\frac{\sqrt{|g|}}{-v_0} \mathrm{d}v^r \wedge \mathrm{d}v^\theta \wedge \mathrm{d}v^\phi \right)}_{\mathrm{d}\mu_{\mathcal{C}_x^+}}.$$

The choice of the volume form $\mathrm{d}\mu_{\mathcal{C}_x^+}$ orients \mathcal{C}_x^+ and, by extension, the set of variables (v^r, v^θ, v^ϕ) used to parametrize \mathcal{C}_x^+ . This induced orientation is the same as the one obtained by contracting $\mathrm{d}\mu_{T_x\mathcal{M}}$ with a future unit normal to \mathcal{C}_x^+ . Far from the horizon, the normal has same direction as ∂_{v^t} . However, ∂_{v^t} is spacelike close to the horizon. The future directed normal to surfaces of constant t is $\partial_t + (2aMr)/\Pi \partial_\phi$, so $\partial_{v^t} + (2aMr)/\Pi \partial_{v^\phi}$ gives a time orientation on \mathcal{C}_x^+ . On some parts of the \mathcal{C}_x^+ , the vector field ∂_t passes through \mathcal{C}_x^+ in the future direction, but in some regions it passes in the past direction. This is illustrated in figure 2.2. Hence, the orientation of the variables (v^r, v^θ, v^ϕ) changes, which compensates for the change of sign of v_0 . Recall v_0 is a linear combination of v^0 and v^1 . The set where v_0 vanishes is of measure 0.

Recall the following (see [4]):

Definition 3. *The Vlasov stress-energy tensor is defined to be*

$$\mathrm{T}_{\alpha\beta}[f]_x = \int_{\mathcal{C}_x^+} f(x, v) v_\alpha v_\beta \mathrm{d}\mu_{\mathcal{C}_x^+}.$$

For the remainder of this paper, the term “stress-energy tensor” will refer to the Vlasov stress-energy tensor. The Vlasov stress-energy tensor is symmetric, traceless, and divergence-free for the massless Vlasov equation. If f is non-negative, the stress-energy tensor satisfies the dominant energy condition.

Killing tensors play a crucial role in understanding the symmetries of the Vlasov equation. Recall $K_{\alpha_1 \dots \alpha_n}$ is a conformal Killing tensor if, for some $n \in \mathbb{N}$, there is a tensor field $p_{\alpha_1 \dots \alpha_{n-1}}$ such that

$$\begin{aligned} K_{\alpha_1 \dots \alpha_n} &= K_{(\alpha_1 \dots \alpha_n)}, \\ \nabla_{(\beta} K_{\alpha_1 \dots \alpha_n)} &= g_{(\beta \alpha_1} p_{\alpha_2 \dots \alpha_n)}. \end{aligned}$$

If K is a conformal Killing tensor, then there are several relevant and well-known consequences. $K_{\alpha_1 \dots \alpha_n} \dot{\gamma}^{\alpha_1} \dots \dot{\gamma}^{\alpha_n}$ is constant along any null geodesic γ . On $T\mathcal{M}$, the function $(x, v) \mapsto K_{\alpha_1 \dots \alpha_n} v^{\alpha_1} \dots v^{\alpha_n}$ is a solution of the Vlasov equation (1.2). Hence, its restriction to \mathcal{C}^+ satisfies the massless Vlasov equation. From this, with the following definition, the subsequent lemma follows by direct calculation.

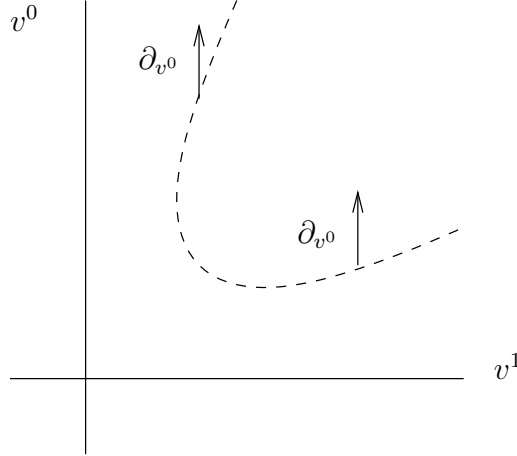


FIGURE 1. If all the ∂_{v^i} are spacelike and the future directed normal to the future part of the cone $g_{\alpha\beta}v^\alpha v^\beta = 0$ lies in the span of ∂_{v^0} and ∂_{v^1} , then ∂_{v^0} is future directed (pointing into the cone) in some regions of the cone but past directed (pointing out of the cone) in others. Here, a hyperboloidal cross-section corresponding to some nonzero (v^2, v^3) is illustrated.

Definition 4. A symmetry of the null Vlasov equation is a linear map S such that, if $f(x, v)$ is a solution of the Vlasov equation, then so is $(Sf)(x, v)$. A multiplicative symmetry is a symmetry that is given by multiplication by a function on the tangent space, i.e. $(Sf)(x, v) = (S(x, v))(f(x, v))$ for some function $S(x, v)$.

Lemma 5. If $K_{\alpha_1 \dots \alpha_n}$ is a conformal Killing tensor, then the map

$$f(x, v) \mapsto K_{\alpha_1 \dots \alpha_n} v^{\alpha_1} \dots v^{\alpha_n} f(x, v)$$

is a multiplicative symmetry of the null Vlasov equation

Lemma 6. Let $n \in \mathbb{N}$, and $\{S_{\underline{a}}\}$ be a collection of multiplicative symmetries. The stress-energy tensor defined by

$$T_{\alpha\beta\underline{a}_1 \dots \underline{a}_n}[f]_x = \int_{\mathcal{C}_x^+} S_{\underline{a}_n} \dots S_{\underline{a}_1} f(x, v) v_\alpha v_\beta d\mu_{\mathcal{C}_x^+}$$

satisfies

- (1) (symmetry) $T_{\alpha\beta\underline{a}_1 \dots \underline{a}_n}[f] = T_{\beta\alpha\underline{a}_1 \dots \underline{a}_n}[f]$,
- (2) (trace-free) $T^\alpha_{\alpha\underline{a}_1 \dots \underline{a}_n}[f] = 0$, and,
- (3) (divergence-free) if f is a solution of the Vlasov equation, $\nabla^\alpha T_{\alpha\beta\underline{a}_1 \dots \underline{a}_n}[f] = 0$.
- (4) (dominant energy condition) Furthermore, suppose Y^α is a future-directed causal vector, and suppose that $X^{\alpha\underline{a}_1 \dots \underline{a}_n}$ is such that for any set of real numbers $\{\sigma_{\underline{a}}\}$, the vector $X^{\alpha\underline{a}_1 \dots \underline{a}_n} \sigma_{\underline{a}_1} \dots \sigma_{\underline{a}_n}$ is a future-directed causal vector. In this case, if f is nonnegative, then $T_{\alpha\beta\underline{a}_1 \dots \underline{a}_n}[f] X^{\alpha\underline{a}_1 \dots \underline{a}_n} Y^\beta \geq 0$.

Remark 7. In the Kerr spacetime, there is the collection of symmetries $\{e^2, el_z, l_z^2, q\}$. If T^α is a future-directed, causal vector field and $\delta^{\underline{a}\underline{b}}$ is the Kronecker δ , then $T^\alpha \delta^{\underline{a}\underline{b}}$ is a

future-directed, causal symmetry-strengthened vector field. The symmetry-strengthened stress-energy tensor $T_{\alpha\beta ab}$ satisfies the dominant energy condition.

Note that if n is odd, then there is no symmetry-strengthened vector field $X^{\alpha a_1 \dots a_n}$ that is future directed and causal in the sense that for all $X^{\alpha a_1 \dots a_n}$, the vector field $X^{\alpha a_1 \dots a_n} \sigma_{a_1} \dots \sigma_{a_n}$ is future directed and causal. Thus, if n is odd, the dominant energy condition follows vacuously.

Proof. For any sequence of values for a_1, \dots, a_n , consider the sequence of concomitants defined by $T_{\alpha\beta a_1 \dots a_k}[f] = T_{\alpha\beta a_1 \dots a_{k-1}}[S_{a_k} f]$. Since $T_{\alpha\beta}$ is symmetric and is trace-free, the $T_{\alpha\beta a_1 \dots a_k}$ have the same property by induction. Similarly, since each S_{a_k} is a symmetry, the $T_{\alpha\beta a_1 \dots a_k}$ is divergence-free for the Vlasov equation by induction.

Suppose the dominant energy condition fails for $T_{\alpha\beta a_1 \dots a_n}$. Thus, there is some smooth $f : \mathcal{C}^+ \rightarrow [0, \infty)$, an $x \in \mathcal{M}$, and $X^{\alpha a_1 \dots a_n}$ and Y^α as in the statement of the theorem such that

$$\int_{\mathcal{C}_x^+} S_{a_n} \dots S_{a_1} f(x, v) v_\alpha v_\beta X^{\alpha a_1 \dots a_n} Y^\beta d\mu_{\mathcal{C}_x^+} < 0.$$

Thus, there is a $w \in T_x \mathcal{M}$ such that $S_{a_n} \dots S_{a_1} f(x, w) w_\alpha w_\beta X^{\alpha a_1 \dots a_n} Y^\beta < 0$. Let σ_a be the value of S_a at (x, w) . (Since the S_a are assumed to be multiplicative symmetry operators depending on (x, v) , this is possible.) Since the S_a are polynomial in the v^a , they are continuous in \mathcal{C}_x^+ . Thus, there is an open neighbourhood W of w in \mathcal{C}_x^+ such that $S_{a_n} \dots S_{a_1} f(x, v) v_\alpha v_\beta X^{\alpha a_1 \dots a_n} Y^\beta < S_{a_n} \dots S_{a_1} f(x, w) w_\alpha w_\beta X^{\alpha a_1 \dots a_n} Y^\beta / 2 < 0$. Let χ be a smooth function on \mathcal{C}_x^+ that is one on an open neighbourhood W' of w and that is supported in W . Thus, $f\chi$ is a non-negative function on \mathcal{C}_x^+ and

$$\begin{aligned} 0 &> \int_{\mathcal{C}_x^+} (\chi(v) f(x, v)) v_\alpha v_\beta (X^{\alpha a_1 \dots a_n} \sigma_{a_n} \dots \sigma_{a_1}) Y^\beta d\mu_{\mathcal{C}_x^+} \\ (2.4) \quad &> \left(\int_{\mathcal{C}_x^+} (\chi(v) f(x, v)) v_\alpha v_\beta d\mu_{\mathcal{C}_x^+} \right) (X^{\alpha a_1 \dots a_n} \sigma_{a_n} \dots \sigma_{a_1}) Y^\beta \end{aligned}$$

Since $X^{\alpha a_1 \dots a_n} \sigma_{a_1} \dots \sigma_{a_n}$ and Y^α are timelike and future-directed vector fields, and $T_{\alpha\beta}$ satisfies the dominant energy condition, it follows that the final term in inequality (2.4) must be nonnegative, which contradicts inequality (2.4). Thus, by contradiction, $T_{\alpha\beta a_1 \dots a_n}$ must satisfy the dominant energy condition. \square

One concludes this section by the standard conservation of energies for Vlasov fields. Let $\{S_a\}_a$ be a collection of symmetries, $X^{a a_1 \dots a_k}$ a collection of vector fields, and Σ be a spacelike hypersurface. The energy of f with respect to the vector X on the hypersurface Σ is

$$E_X[f](\Sigma) = \int_{\Sigma} T_{\alpha\beta a_1 \dots a_k}[f] X^{\alpha a_1 \dots a_k} d\nu_{\Sigma}^\beta,$$

Let now Σ_1, Σ_2 be hypersurfaces and R be an open set such that $\partial R = \Sigma_2 - \Sigma_1$. The following lemma states the conservation of energies of Vlasov fields:

Lemma 8. *Let Ω be a positive function on \mathcal{M} , and $q^{a_1 \dots a_k}$ be a collection of functions on \mathcal{M} . The following identity holds:*

$$E_X[f](\Sigma_2) - E_X[f](\Sigma_1) = \int_R \Pi_{X,\Omega,q}[f] d\mu_g,$$

where

$$\Pi_{X,\Omega,q}[f] = -\frac{1}{2}\Omega^2 T_{\alpha\beta a_1 \dots a_k}[f] \text{Lie}_{X^{a_1 \dots a_k}}(\Omega^{-2}g^{\alpha\beta}) + T_{\alpha\beta a_1 \dots a_k}g^{\alpha\beta}[f]q^{a_1 \dots a_k}.$$

Remark 9. *Later in Section 3, to simplify the notations, $\Pi_{X,\Omega,q}[f]$ is sometimes denoted Π_X , since Ω and q are clear from context.*

Proof. The proof is a straightforward consequence of the fact that $T_{\alpha\beta a_1 \dots a_k}[f]$ is traceless and divergence free. \square

3. THE BOUNDED-ENERGY ESTIMATE

3.1. Strategy of the argument. In this subsection, we outline the strategy of the rest of this section. For simplicity, we suppress Ω and q , and we initially conflate vector fields with symmetry-strengthened vector fields, the latter of which are actually collections of vector fields. Part of the argument will involve a choice of cut-off function χ , which is used in the construction of a symmetry-strengthened vector field \mathbf{T}_χ .

The essential parts of the proof are to construct symmetry-strengthened vector fields \mathbf{T}_χ and \mathbf{A} such that, for nonzero solutions of the null Vlasov equation,

$$(3.1a) \quad E_{\mathbf{T}_\chi} \geq 0,$$

$$(3.1b) \quad \Pi_{\mathbf{A}} > 0,$$

$$(3.1c) \quad E_{\mathbf{T}_\chi} \gtrsim |E_{\mathbf{A}}|,$$

$$(3.1d) \quad \frac{|a|}{M} \Pi_{\mathbf{A}} \gtrsim |\Pi_{\mathbf{T}_\chi}|.$$

Subsection 3.5 shows that, for sufficiently small $|a|/M$, one can quickly go from bounds like estimates (3.1) to energy and Morawetz estimates of the form $E_{\mathbf{T}_\chi}(\Sigma_{t_2}) \lesssim E_{\mathbf{T}_\chi}(\Sigma_{t_1})$ and $\int_{t_1}^{t_2} \int_{\Sigma_t} \Pi_{\mathbf{A}} \lesssim E_{\mathbf{T}_\chi}(\Sigma_{t_1})$ for all $t_1, t_2 \in \mathbb{R}$.

Of the properties above, the first property (3.1a) is ensured by taking \mathbf{T}_χ to be future-directed and causal. The second property (3.1b) is ensured by finding \mathbf{A} (together with a conformal factor and a collection of auxiliary function q) such that (suppressing symmetry indices for simplicity)

$$(3.2) \quad \Omega^{-2} \Pi_{\mathbf{A},\Omega,q} = \left(-\frac{1}{2} \text{Lie}_{\mathbf{A}}(\Omega^{-2}g^{\alpha\beta}) + q\Omega^{-2}g^{\alpha\beta} \right) T_{\alpha\beta}$$

is non-negative. The properties (3.1c)-(3.1d) are more refined versions of properties (3.1a)-(3.1b). Condition (3.1c) holds for vector fields \mathbf{T}_χ and \mathbf{A} by the dominant energy condition, if \mathbf{A} always lies in the orthogonal hyperplane to \mathbf{T}_χ and the asymptotics of \mathbf{A} are such that it always has length, in that hyperplane, bounded by the length of \mathbf{T}_χ . Condition (3.1d) would hold trivially if \mathbf{T}_χ were Killing, since the associated bulk term would vanish. Because the Kerr exterior has no globally Killing, causal vector,

we instead use the cut-off function χ to construct an approximately Killing vector field. The quantity $|a|/M$ is a measure of the failure of \mathbf{T}_χ to be Killing.

Ideally, one would construct \mathbf{A} and \mathbf{T}_χ that are vector fields, but, following [3], we take them to be symmetry-strengthened vector fields. The Kerr spacetime has orbits, which we define to be null geodesics that neither are absorbed through the event horizon nor escape to null infinity. It is a property of the Kerr spacetime that the orbits remain on surfaces of constant r . The projection of the orbits to Σ_t fills an open set in the Kerr spacetime, but not its tangent space. Because of the presence of orbits in an open set, it is not possible to find a vector field \mathbf{A} such that the term in formula (3.2) is non-negative; however, although [3] doesn't use the terminology introduced in this paper, it introduced symmetry-strengthened vector fields that, with the energies and bulk terms for the wave equation, satisfy conditions (3.1). Most of the rest of this paper consists of constructing these symmetry-strengthened vector fields and demonstrating they have the desired properties. In [3], it was important to work with symmetry-strengthened vector fields so that the quadratic stress-energy tensor for the wave equation could be written as a bilinear quantity. In this paper, it is again convenient to work with symmetry-strengthened vector fields, so that we can more easily define the notion of a causal symmetry-strengthened vector field.

The calculations in this paper are significantly simpler than in [3]. Both papers rely on properties of null geodesics and on the fact that, for a geodesic γ with the energies and bulk terms defined by $E_X[\gamma] = \dot{\gamma}^\alpha X_\alpha$ and $\Pi_X[\gamma] = \nabla_{(\alpha} X_{\beta)} \dot{\gamma}^\alpha \dot{\gamma}^\beta$, the estimates (3.1a)-(3.1d) are valid. The calculations in this paper are relatively quick, since the behaviour of null geodesics completely determines the behaviour of solutions to the Vlasov equation. In contrast, solutions of the wave equation are only accurately modelled by null geodesics in the high-frequency limit. To treat the wave equation in [3], a significant amount of additional work is required to show that a similar method can be used uniformly without a frequency decomposition.

3.2. The blended energy. In this subsection, we construct a causal symmetry-strengthened vector field.

Definition 10. *Let*

$$\begin{aligned} T_\perp &= \left(\partial_t + \frac{2aMr}{\Pi} \partial_\phi \right)^\alpha = (\partial_t + \omega_\perp \partial_\phi)^\alpha, \\ T_\chi^\alpha &= (\partial_t + \chi \omega_\mathcal{H} \partial_\phi)^\alpha, \\ \mathbf{T}_\chi^{\alpha\bar{a}\bar{b}} &= T_\chi^\alpha \delta^{\bar{a}\bar{b}}, \end{aligned}$$

where $\omega_\mathcal{H} = a/(r_+^2 + a^2)$ is the rotation speed of the horizon, $\chi = \chi(r)$ is a function that is 1 for $r < r_\chi$, smoothly decreasing on $r \in [r_\chi, r_\chi + M]$, and identically 0 for $r > r_\chi + M$, and where r_χ is chosen sufficiently large. For simplicity, we take $r_\chi = 10M$.

Lemma 11. *There is a positive constant $\bar{\epsilon}$ such that if $|a| < \bar{\epsilon}M$, $t \in \mathbb{R}$, and $f : \mathcal{C}^+ \rightarrow [0, \infty)$ is continuous, then*

$$(3.3) \quad E_{T_\perp}[f](\Sigma_t) \simeq \int_{\Sigma_t} \int_{\mathcal{C}_x^+} \left(\frac{(r^2 + a^2)^2}{\Delta} v_t^2 + \Delta v_r^2 + Q^{\alpha\beta} v_\alpha v_\beta \right) f d\mu_{\mathcal{C}_x^+} d\mu_{\Sigma_t},$$

$$(3.4) \quad \simeq \int_{\Sigma_t} \int_{\mathcal{C}_x^+} \left(\frac{(r^2 + a^2)^2}{\Delta} v_t^2 + \Delta v_r^2 + v_\theta^2 + \frac{1}{\sin^2 \theta} v_\phi^2 \right) f d\mu_{\mathcal{C}_x^+} d\mu_{\Sigma_t},$$

$$(3.5) \quad E_{T_\perp}[f](\Sigma_t) \simeq E_{T_\chi}[f](\Sigma_t),$$

$$(3.6) \quad \simeq E_{\text{model},3}.$$

Furthermore, if f is a C^1 solution of the Vlasov equation and $\Omega^{-2} = \Sigma$, then

$$(3.7) \quad \sqrt{\det g} \left| -\frac{1}{2} \Omega^2 T[f]_{\alpha\beta} \text{Lie}_{T_\chi}(\Omega^{-2} g^{\alpha\beta}) \right| = \Delta |\partial_r \chi| |v_r| |v_\phi| \sin \theta.$$

Proof. This proof follows the argument of the proof for Lemma 3.1 of [3].

Let ω_\perp denote $2aMr/\Pi$. Since the normal satisfies $dv_{\Sigma_t}^\alpha = T_\perp^\alpha(\Pi/\Delta) dr d\theta d\phi$, and since $-g_{\alpha\beta} T_\perp^\alpha T_\perp^\beta = \Delta \Sigma / \Pi$, the T_\perp energy is

$$E_{T_\perp} = \int_{\Sigma_t} T_{\alpha\beta} T_\perp^\alpha T_\perp^\beta \frac{\Pi}{\Delta} dr d\theta d\phi = \int_{\Sigma_t} \int_{\mathcal{C}_x^+} f \left(\frac{\Pi}{\Delta} (T_\perp^\alpha v_\alpha)^2 + \frac{1}{2} \Sigma g^{\alpha\beta} v_\alpha v_\beta \right) d\mu_{\mathcal{C}_x^+} dr d\theta d\phi.$$

The integrand can be expanded as

$$(3.8a) \quad \frac{\Pi}{\Delta} (T_\perp^\alpha v_\alpha)^2 + \frac{1}{2} \Sigma g^{\alpha\beta} v_\alpha v_\beta = \frac{1}{2} \left(\Delta (v_r)^2 + \frac{(r^2 + a^2)^2}{\Delta} (T_\perp^\alpha v_\alpha)^2 + Q^{\alpha\beta} v_\alpha v_\beta + v_\phi^2 \right)$$

$$(3.8b) \quad - \frac{1}{2\Delta} (4aMr - 2\omega_\perp (r^2 + a^2)^2) v_t v_\phi$$

$$(3.8c) \quad + \frac{1}{2\Delta} (-a^2 + (r^2 + a^2)^2 \omega_\perp^2) v_\phi^2 - a^2 \sin^2 \theta (T_\perp^\alpha v_\alpha)^2.$$

Since the coefficients $4aMr - 2\omega_\perp (r^2 + a^2)^2$ and $-a^2 + (r^2 + a^2)^2 \omega_\perp^2$ vanish at $r = r_+$, are bounded by factors that go uniformly to 0 on bounded sets as $a \rightarrow 0$, and grow as $r \rightarrow \infty$ no faster than r and a constant respectively, for $|a|$ sufficiently small, the terms in lines (3.8b)-(3.8c) are dominated by those on the right-hand side of line (3.8a). Thus, the terms on the left and right side of line (3.8a) are equivalent. This proves estimate (3.3). Estimate (3.4) follows from the equivalence

$$(T_\perp^\alpha v_\alpha)^2 + Q^{\alpha\beta} v_\alpha v_\beta \simeq (T_\perp^\alpha v_\alpha)^2 + v_\theta^2 + \frac{1}{\sin^2 \theta} v_\phi^2.$$

The T_χ energy can be estimated using the fact that $T_\perp - T_\chi = (\omega_\perp - \chi \omega_{\mathcal{H}}) \partial_\phi$ is orthogonal to T_\perp , so

$$E_{T_\perp} - E_{T_\chi} = \int_{\Sigma_t} (\omega_\perp - \chi \omega_{\mathcal{H}}) v_\phi (T_\perp^\alpha v_\alpha) \frac{\Pi}{\Delta} d\mu_{\Sigma_t}.$$

The coefficient $\omega_\perp - \chi \omega_{\mathcal{H}}$ vanishes linearly at $r = r_+$, is bounded by a function that goes to zero uniformly as $a \rightarrow 0$, and goes to zero as $r \rightarrow \infty$ like r^{-4} , so, by a simple Cauchy-Schwarz estimate, one finds $|E_{T_\perp} - E_{T_\chi}| \lesssim |a| E_{T_\perp}$, and $E_{T_\perp} \simeq E_{T_\chi}$. Finally, E_{T_\perp} and $E_{\text{model},3}$ are equivalent, since, in considering the integration on the cone, $\sqrt{\det g} = \Sigma \sin \theta$ is uniformly equivalent to $r^2 \sin \theta$ for a sufficiently small.

The contraction of the stress-energy tensor with the Lie derivative can be calculated directly from $\text{Lie}_{T_\chi}(\Omega^{-2}g^{\alpha\beta}) = -2\Delta\partial_r^{(\alpha}\partial_\phi^{\beta)}$. \square

Corollary 12. *There is a positive constant $\bar{\epsilon}$ such that if $|a| \leq \bar{\epsilon}$, $t \in \mathbb{R}$, and $f : \mathcal{C}^+ \rightarrow [0, \infty)$ is continuous, then*

$$E_{\mathbf{T}_\chi}[f](\Sigma(t)) \simeq E_{\text{model},3}[f](t).$$

Proof. This follows from applying estimates (3.4) and (3.5), substituting $(M^2v_t^2 + v_\theta^2 + \csc^2\theta v_\phi^2)^2 f$ for f , recognising $E_{\mathbf{T}_\chi}[f]$ as being obtained from $E_{T_\chi}[f]$ substituting $(M^2v_t^2 + q + l_z^2)^2 f$ for f , and observing the uniform equivalence of $(M^2v_t^2 + q + l_z^2)$ and $(M^2v_t^2 + v_\theta^2 + \csc^2\theta v_\phi^2)$. \square

3.3. Set-up for radial vector fields. In this subsection, we define a radial symmetry-strengthened vector field, \mathbf{A} , in terms of unspecified scalar functions, which will be chosen in the following subsection. The main result of this subsection is that the bulk term, $\Pi_{\mathbf{A}}$, can be written as a sum of two terms, with the second involving a square and the first involving a second derivative. A square is always non-negative. One should expect that the second-derivative term will be non-negative on orbits, since the orbits are known to be unstable. In the following subsection, the scalar functions are chosen so that this second-derivative term is non-negative everywhere, not just on the orbits.

Definition 13. *If z and w are smooth functions of r and the parameters M and a , the Morawetz symmetry-strengthened vector field and the reduced scalar functions are defined to be*

$$\begin{aligned} \mathbf{A}^{\alpha\bar{a}\bar{b}} &= -zw\mathcal{L}^{(a}\tilde{\mathcal{R}}^{\bar{b})}\partial_r^\alpha, \\ q^{\bar{a}\bar{b}} &= \frac{1}{2}(\partial_r z)w\mathcal{L}^{(a}\tilde{\mathcal{R}}^{\bar{b})}, \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathcal{R}}'^{\bar{a}} &= \partial_r \left(\frac{z}{\Delta} \mathcal{R}^{\bar{a}} \right), \\ \mathcal{L} &= \mathcal{L}^{\bar{a}} S_{\bar{a}} = M^2 e^2 + l_z^2 + q, \end{aligned}$$

and $\mathcal{R}^{\bar{a}}$ and \mathcal{L} are defined in equations (2.2) and (2.3). We also introduce

$$\tilde{\mathcal{R}}'' = \partial_r \left(w \frac{z^{1/2}}{\Delta^{1/2}} \tilde{\mathcal{R}}' \right).$$

The following lemma is a trivial observation in the current context. This is in contrast with the situation for the wave equation where the rearrangement of the symmetry indices required some calculation and introduced additional terms at the initial and final time, which had to be dominated by the energies.

Lemma 14 (Rearrangements).

$$\begin{aligned} \mathcal{L}^{(a}\mathcal{R}^{\bar{b})} S_{\bar{a}} S_{\bar{b}} &= \mathcal{L}^{(a}\mathcal{R}^{\bar{b})} S_{\bar{a}}^{\alpha\beta} S_{\bar{b}}^{\gamma\delta} v_\alpha v_\beta v_\gamma v_\delta \\ &= \mathcal{L}^{\bar{a}} \mathcal{R}^{\bar{b}} S_{\bar{a}}^{\alpha\beta} S_{\bar{b}}^{\gamma\delta} v_\alpha v_\beta v_\gamma v_\delta. \end{aligned}$$

Proof. Apply the definition $S_{\underline{a}} = S_{\underline{a}}^{\alpha\beta} v_\alpha v_\beta$ and similarly in \underline{b} , and then observe that the contraction in $\alpha\beta\gamma\delta$ is against four copies of v , so that it is automatically symmetric in \underline{ab} . \square

Lemma 15. *With \mathbf{A} and q as above and $\Omega^{-2} = \Sigma$, one finds*

$$\begin{aligned} \Pi_{\mathbf{A},\Omega,q} &= \mathcal{L}^{\underline{a}} \left(-z^{1/2} \Delta^{3/2} \partial_r \left(w \frac{z^{1/2}}{\Delta^{1/2}} \tilde{\mathcal{R}}'^{\underline{b}} \right) \partial_r^\epsilon \partial_r^\lambda + \frac{1}{2} w \tilde{\mathcal{R}}'^{\underline{b}} \tilde{\mathcal{R}}'^{\underline{c}} S_{\underline{c}}^{\epsilon\lambda} \right) S_{\underline{a}}^{\alpha\beta} S_{\underline{b}}^{\gamma\delta} v_\alpha v_\beta v_\gamma v_\delta v_\epsilon v_\lambda f. \end{aligned}$$

Proof. Recall

$$\begin{aligned} \Omega^{-2} g^{\epsilon\lambda} &= \Delta \partial_r^\epsilon \partial_r^\lambda + \frac{1}{\Delta} \mathcal{R}^{\epsilon\lambda}, \\ \Omega^{-2} \Pi_{\mathbf{A},\Omega,q} &= -\frac{1}{2} \text{Lie}_{\mathbf{A}^{ab}} (\Omega^{-2} g^{\alpha\beta}) + \Omega^{-2} q^{ab} g^{\alpha\beta}. \end{aligned}$$

Thus,

$$\begin{aligned} \Omega^{-2} \Pi_{\mathbf{A},\Omega,q} &= \left(\frac{1}{2} \left(\mathcal{L}^{(\underline{a}} z w \tilde{\mathcal{R}}'^{\underline{b}}) \partial_r \Delta - 2 \Delta \mathcal{L}^{(\underline{a}} \partial_r (z w \tilde{\mathcal{R}}'^{\underline{b}}) \right) \partial_r^\epsilon \partial_r^\lambda + \frac{1}{2} \mathcal{L}^{(\underline{a}} z w \tilde{\mathcal{R}}'^{\underline{b}}) \partial_r \left(\frac{\mathcal{R}^{\epsilon\lambda}}{\Delta} \right) \right. \\ &\quad \left. + \frac{1}{2} (\partial_r z) w \mathcal{L}^{(\underline{a}} \tilde{\mathcal{R}}'^{\underline{b}}) \partial_r^\epsilon \partial_r^\lambda + \frac{1}{2} (\partial_r z) w \mathcal{L}^{(\underline{a}} \tilde{\mathcal{R}}'^{\underline{b}}) \frac{1}{\Delta} \mathcal{R}^{\epsilon\lambda} \right) \\ &\quad S_{\underline{a}}^{\alpha\beta} S_{\underline{b}}^{\gamma\delta} v_\alpha v_\beta v_\gamma v_\delta v_\epsilon v_\lambda \\ &= \mathcal{L}^{\underline{a}} \left(-z^{1/2} \Delta^{3/2} \partial_r \left(w \frac{z^{1/2}}{\Delta^{1/2}} \tilde{\mathcal{R}}'^{\underline{b}} \right) \partial_r^\epsilon \partial_r^\lambda + \frac{1}{2} w \tilde{\mathcal{R}}'^{\underline{b}} \partial_r \left(\frac{z}{\Delta} \mathcal{R}^{\epsilon\lambda} \right) \right) \\ &\quad S_{\underline{a}}^{\alpha\beta} S_{\underline{b}}^{\gamma\delta} v_\alpha v_\beta v_\gamma v_\delta v_\epsilon v_\lambda. \end{aligned}$$

Substituting $\tilde{\mathcal{R}}'^{\epsilon\lambda} = \partial_r \left(\frac{z}{\Delta} \mathcal{R}^{\epsilon\lambda} \right)$ gives the desired result. \square

3.4. Choosing the weights. In this subsection, we choose the weights z and w , so that $\Pi_{\mathbf{A},\Omega,q}$ is non-negative for all r . The choices are the same as those appearing for the wave equation in [3].

Here, we recall how the weight functions z and w are chosen, following the explanation in remark 3.8 of [3]. The goal in choosing the various weight functions is to obtain nonnegativity for the two terms in $\Pi_{\mathbf{A},\Omega,q}$, namely $-z^{1/2} \Delta^{3/2} \partial_r \left(w \frac{z^{1/2}}{\Delta^{1/2}} \tilde{\mathcal{R}}' \right) v_r^2$ and $\frac{1}{2} w \tilde{\mathcal{R}}' \tilde{\mathcal{R}}'$. For $|a| \ll M$, the orbiting null geodesics are near $r = 3M$. On orbiting null geodesics, $\tilde{\mathcal{R}}'(r; M, a; e, l_z, q)$ vanishes and $-\tilde{\mathcal{R}}''(r; M, a; e, l_z, q)$ is positive. Thus, the desired non-negativity holds on orbiting null geodesics regardless of the choice of z and w . The functions z and w are chosen so that the non-negativity extends to all other null geodesics. These functions can be chosen so that $-\tilde{\mathcal{R}}''(r; M, a; e, l_z, q)$ remains positive everywhere and so that $\tilde{\mathcal{R}}'(r; M, a; e, l_z, q)$ vanishes only in a neighbourhood of $r = 3M$.

We choose the weights so that the following properties hold:

- (1) The definition of $\tilde{\mathcal{R}}'$ in equation (3.9) is made so that $w\tilde{\mathcal{R}}'\partial_r(\frac{z}{\Delta}\mathcal{R})$ takes the form $w\tilde{\mathcal{R}}'^2$ in Lemma 15.
- (2) $M^2\epsilon_{e^2}^2$ is the coefficient of e^2 in $\tilde{\mathcal{R}}'(r; M, a; e, l_z, q)$ and $\tilde{\mathcal{R}}''(r; M, a; e, l_z, q)$. Note that $M^2\epsilon_{e^2}^2$ plays the same role as $\epsilon_{\partial_t}^2$ in [3], where the differential symmetry operator ∂_t^2 for the wave equation plays the role of the multiplicative symmetry e^2 for the Vlasov equation. The use of a dimensionless parameter, ϵ_{e^2} , in this paper clarifies that the small parameter $|a|/M$ can be chosen uniformly in M .
- (3) z_1 is such that, if z_2 had been equal to 1, which corresponds to $\epsilon_{e^2} = 0$, then the coefficient of M^2e^2 in $\tilde{\mathcal{R}}'(r; M, a; e, l_z, q)$ would be zero.
- (4) z_2 is such that, if $\epsilon_{e^2} > 0$, then the coefficient of $M^2\epsilon_{e^2}e^2$ in $\tilde{\mathcal{R}}'(r; M, a; e, l_z, q)$ is non-negative and a perturbation (in ϵ_{e^2}) of the coefficient of q .
- (5) w_1 is such that, if z_2 and w_2 had both been equal to 1, then the coefficient of $Me l_z$ in $\tilde{\mathcal{R}}''(r; M, a; e, l_z, q)$ would vanish.
- (6) w_2 is such that
 - (a) $\tilde{\mathcal{R}}''(r; M, a; e, l_z, q)$ is positive everywhere, and
 - (b) $(zw\tilde{\mathcal{R}}'(r; M, a; e, l_z, q))^2g(\partial_r, \partial_r) \lesssim -(M^2e^2 + l_z^2 + q)^2g(T_\chi, T_\chi)$.

In particular, from the dominant energy condition, condition 6b allows us to show that $|E_{\mathbf{A}}| \lesssim E_{\mathbf{T}_\chi}$. (However, in the proof of Lemma 18, it is easier to apply the Cauchy-Schwarz inequality directly, rather than to work through subtle details in applying the dominant energy condition.) Once the form $w_2 = Cr^{-1}$ was chosen, the factor of $C = 1/2$ was chosen so that, when $a = 0$ and $\epsilon_{e^2} = 0$, the coefficient of $l_z^2 + q$ in $\tilde{\mathcal{R}}''$ is equal to 1.

The factors $\tilde{\mathcal{R}}'$, z_1 , z_2 , and w_1 are uniquely defined by the above properties. In contrast, the factor w_2 is both overdetermined, since we have chosen it to satisfy two conditions that are not a priori obviously compatible, and underdetermined, since it so happens that there are many functions that allow these two conditions to be satisfied.

Definition 16. *Given a positive value for the parameter ϵ_{e^2} , we use the following weights to define the Morawetz vector field,*

$$\begin{aligned}
 z &= z_1 z_2, & w &= w_1 w_2, \\
 z_1 &= \frac{\Delta}{(r^2 + a^2)^2}, & w_1 &= \frac{(r^2 + a^2)^4}{3r^2 - a^2}, \\
 z_2 &= 1 - M^2\epsilon_{e^2} \frac{\Delta}{(r^2 + a^2)^2}, & w_2 &= \frac{1}{2r}.
 \end{aligned}$$

The reason for these choices is explained in Remark 3.8 of [3].

In the following lemma, big- O notation is used in the r variable. The notation $f = O(r^{-l})$ means that f is independent of v_r , e , l_z , and q and that there is a constant C such that for positive M and sufficiently small $|a|/M$, uniformly in $r > r_+$, there is the bound $|f(r, M, a)| \leq Cr^{-l}$. The notation $f = g + hO(r^{-l})$ denotes that there is a function $k = k(r, M, a)$ such that $k = O(r^{-l})$. The notation $f = g + h_1O(r^{-l_1}) + \dots + h_nO(r^{-l_n})$ is defined recursively.

Lemma 17. *With \mathbf{A} and q as above and $\Omega^{-2} = \Sigma$, there are positive constants $\bar{\epsilon}$, ϵ_{e2} , and C such that if $|a| \leq \bar{\epsilon}M$, $0 < \epsilon_{e2} \leq \epsilon_{e2}^-$ and $f : \mathcal{C}^+ \rightarrow [0, \infty)$ is a solution of the Vlasov equation, then*

$$(3.9) \quad C\Omega^2\Pi_{\mathbf{A}} \geq M \frac{\Delta^2}{(r^2 + a^2)^2} v_r^2 |f|_2 + r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L}f.$$

and

$$(3.10) \quad \begin{aligned} \tilde{\mathcal{R}}' = & -2r^{-4}(r - 3M)\mathcal{L}_{\epsilon_{e2}} \\ & + aMO(r^{-4})el_z \\ & + a^2(O(r^{-5})q + O(r^{-5})l_z^2) \\ & + M^2\epsilon_{e2}(a^2O(r^{-5})e^2 + O(r^{-5})q + O(r^{-5})l_z^2). \end{aligned}$$

Proof. Direct calculation of $\tilde{\mathcal{R}}'$ with our choices of z and w gives

$$\begin{aligned} \tilde{\mathcal{R}}' = & -M^2\epsilon_{e2}(2(r - 3M)r^{-4} + a^2O(r^{-5}))e^2 \\ & + aMO(r^{-4})el_z \\ & - (2(r - 3M)r^{-4} + a^2O(r^{-5}) + M^2\epsilon_{e2}O(r^{-5}))q \\ & - (2(r - 3M)r^{-4} + a^2O(r^{-5}) + M^2\epsilon_{e2}O(r^{-5}))l_z^2. \end{aligned}$$

Grouping the terms in orders of ϵ_{e2} and a gives equation (3.10). From this,

$$\begin{aligned} -\tilde{\mathcal{R}}'' = & M^3\epsilon_{e2}(r^{-2} + a^2O(r^{-3}) + M^2\epsilon_{e2}O(r^{-3}))e^2 \\ & + aMO(r^{-2})el_z \\ & + M(r^{-2} + a^2O(r^{-3}) + M^2\epsilon_{e2}O(r^{-3}))q \\ & + M(r^{-2} + a^2O(r^{-3}) + M^2\epsilon_{e2}O(r^{-3}))l_z^2. \end{aligned}$$

In the coefficients of $M^2\epsilon_{e2}e^2$, q , and l_z^2 , the Mr^{-2} term dominates the remaining terms for sufficiently small $|a|$ and ϵ_{e2} . From the Cauchy-Schwarz inequality, the el_z term is dominated by the e^2 and l_z^2 terms for sufficiently small $|a|$. Thus, fixing ϵ_{e2} sufficiently small and choosing a constant accordingly,

$$-\tilde{\mathcal{R}}'' \geq CM(r^2 + a^2)^{-1}(M^2e^2 + q + l_z^2).$$

Thus, for $|a|$ sufficiently small and ϵ_{e2} as above,

$$\Omega^2\Pi_{\mathbf{A}} \geq CM \frac{\Delta^2}{(r^2 + a^2)^2} v_r^2 |f|_2 + \frac{1}{4r} \frac{(r^2 + a^2)^4}{3r^2 - a^2} \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L}f.$$

Thus, there is a new constant C , such that

$$C\Omega^2\Pi_{\mathbf{A}} \geq M \frac{\Delta^2}{(r^2 + a^2)^2} v_r^2 |f|_2 + r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L}f.$$

□

Lemma 18 (Controlling the boundary terms). *With \mathbf{A} and q as above and $\Omega^{-2} = \Sigma$, and with ϵ_{e2} as in Lemma 17, there is a constant C such that for any $f : \mathcal{C}^+ \rightarrow \mathbb{R}$ and $t \in \mathbb{R}$,*

$$|E_{\mathbf{A}}[f](\Sigma_t)| \leq CE_{\mathbf{T}_\chi}[f](\Sigma_t).$$

Proof. This follows from lemma 3.11 of [3]. Assume both energies are defined. By direct computation,

$$\begin{aligned} E_{\mathbf{A}} &= - \int_{\Sigma_t} \left(T_{\alpha\beta\bar{a}\bar{b}} \mathbf{A}^{\alpha\bar{a}\bar{b}} \right)_{\alpha} T_{\perp}^{\alpha} \frac{\Pi}{\Delta} \sin \theta dr d\theta d\phi, \\ |E_{\mathbf{A}}| &\leq C \int_{\Sigma_t} \left(|T_{\perp}^{\alpha} v_{\alpha}| |\mathbf{A}^{r\alpha\beta}| |S_{\bar{a}} S_{\bar{b}}| |v_r| \frac{\Pi}{\Delta} \right) \sin \theta dr d\theta d\phi \\ &\leq C \int_{\Sigma_t} \left(\frac{\Pi}{\Delta} |T_{\perp}^{\alpha} v_{\alpha}|_2^2 + \frac{\Pi}{\Delta} \left(\sum_{\bar{a}, \bar{b}} |\mathbf{A}^{r\bar{a}\bar{b}}|^2 \right) |v_r|_2^2 \right) \sin \theta dr d\theta d\phi. \end{aligned}$$

Since Π/Δ , $(r^2 + a^2)^2/\Delta$, and r^4/Δ are all uniformly equivalent and since $\sum_{\bar{a}, \bar{b}} \mathbf{A}^{r\bar{a}\bar{b}}$ is bounded by a multiple of Δr^{-2} , it follows from estimate (3.4) that $|E_{\mathbf{A}}| \leq CE_{\mathbf{T}_\chi}$. Since this bound followed from the Cauchy-Schwarz inequality, if $E_{\mathbf{T}_\chi}$ is finite, then the absolute value of the integrand in $E_{\mathbf{A}}$ is integrable, and $E_{\mathbf{A}}$. If $E_{\mathbf{T}_\chi}$ is infinite, then the desired estimate holds trivially. Thus, the initial assumption that both energies are finite is redundant. \square

3.5. Closing the argument.

Proof of Theorems 1 and 2. Let $t_1, t_2 \in \mathbb{R}$. Initially, assume that f restricted to $\pi^{-1}(\Sigma_{t_1})$ has compact support in $\pi^{-1}(\Sigma_{t_1})$. By standard results for the Vlasov equation, this means f restricted to $\pi^{-1}(\Sigma_t)$ has compact support in each $\pi^{-1}(\Sigma_t)$. From integrating the result of Lemma 17, one finds

$$\begin{aligned} E_{\mathbf{A}}[f](t_2) - E_{\mathbf{A}}[f](t_1) \\ \geq \int_{t_1}^{t_2} \int_{\Sigma_t} \int_{\mathcal{C}^+} \left(M \frac{\Delta^2}{(r^2 + a^2)^2} v_r^2 |f|_2 + r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L} f \right) d\mu_{\mathcal{C}_x^+} d\mu_g. \end{aligned}$$

Applying Lemma 18, one finds

$$\begin{aligned} (3.11) \quad &E_{\mathbf{T}_\chi}[f](t_2) + E_{\mathbf{T}_\chi}[f](t_1) \\ &\geq C \int_{t_1}^{t_2} \int_{\Sigma_t} \int_{\mathcal{C}^+} \left(M \frac{\Delta^2}{(r^2 + a^2)^2} v_r^2 |f|_2 + r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L} f \right) d\mu_{\mathcal{C}_x^+} d\mu_g. \end{aligned}$$

From multiplying equation (3.7) for Π_{T_χ} by $(M^2 v_t^2 + q + l_z^2)^2$, one obtains the bulk term for $\Pi_{\mathbf{T}_\chi}$. Integrating this over $\bigcup_{t \in [t_1, t_2]} \Sigma_t$, and observing that $|\partial_r \chi|$ is compactly

supported and that $\omega_{\mathcal{H}}$ vanishes linearly in a , one finds

$$\begin{aligned}
 & E_{\mathbf{T}_\chi}[f](t_2) - E_{\mathbf{T}_\chi}[f](t_1) \\
 & \leq \int_{t_1}^{t_2} \int_{\Sigma_t} \int_{\mathcal{C}_x^+} (M^2 v_t^2 + q + l_z^2)^2 \Delta |\partial_r \chi| |v_r| |v_\phi| f d\mu_{\mathcal{C}_x^+} \Sigma^{-1} d\mu_g \\
 (3.12) \quad & \leq \frac{|a|}{M} C \int_{t_1}^{t_2} \int_{\Sigma_t} \int_{\mathcal{C}^+} M \frac{\Delta^2}{(r^2 + a^2)^2} v_r^2 |f|_2 + r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L} f d\mu_{\mathcal{C}_x^+} \Sigma^{-1} d\mu_g.
 \end{aligned}$$

Combining equations (3.11) and (3.12) and taking $|a|/M$ sufficiently small, one finds that there is a constant C such that

$$E_{\mathbf{T}_\chi}[f](\Sigma_{t_2}) \leq C E_{\mathbf{T}_\chi}[f](\Sigma_{t_1}).$$

Taking $t_2 = t$ and $t_1 = 0$ proves Theorem 1 for solutions with compactly supported data. From this, Estimate (3.11), and taking the limits $t_2 \rightarrow \infty$ with $t_1 = 0$ and $t_1 \rightarrow -\infty$ with $t_2 = 0$, one finds equation (1.6). Observing that $\tilde{\mathcal{R}}'$ grows like $O(r^{-3})(M^2 e^2 q + l_z^2)$ for large r and has a simple root near $r = 3M$ allows us to replace $r^5 \tilde{\mathcal{R}}' \tilde{\mathcal{R}}' \mathcal{L} f$ by $r^{-1} \mathbf{1}_{r \neq 3M} (M v_t^2 + v_\theta^2 + v_\phi^2) |f|_2$. This proves estimate (1.5) and completes the proof of Theorem 2 for solutions with compactly supported data. Since the bounds do not depend on the support of the initial data, by density Theorems 1 and 2 hold for all functions for which $E_{\mathbf{T}_\chi}$ is finite. The theorems follow trivially when this energy is infinite. \square

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